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Mott scattering spin measurement as a generalised quantum measurement

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Abstract. We show that the change of states in the Mott scattering spin measurement, in the spin factor space, belongs to the class of generalised quantum measurements. An explicit example of the identity decomposition, which corresponds to the Mott scattering spin measurement, is given.

1. Introduction

The aim of this paper is to compare the Stern-Gerlach spin measurement (SGSM) and the Mott scattering spin measurement (MSSM) in order to show that the MSSM belongs to the generalised quantum measurements (Kraus 1983).

The standard quantum measurement, described by the well known 'projection postulate' (von Neumann 1956), contains two assumptions that can hardly be avoided in any description of a measurement: the first one is that the macroscopically different events are represented by orthogonal vectors and states (in the chosen Hilbert space), and the second one is the repeatability hypothesis. Nevertheless, the projection postulate gives a highly idealised description of the measurement process. In particular, it is incompatible with the quantum dynamical law and, what is very important, the most frequent and perhaps the only possible measurement, that of the position, cannot be properly incorporated in the standard quantum description (see, e.g., Davies 1976 and Ozawa 1984). Some important developments aimed at a more realistic description of the measurement process have been published recently (e.g. by Ghirardi *et al* (1986), Joos and Zeh (1985) and Kraus (1983)) and, independently of their slight differences, their common characteristic is the use of the framework of generalised dynamical maps (Sudarshan *et al* 1961) in which generalised measurements belong to the class of completely positive dynamical maps.

The aim of this paper is to show that the MSSM (Kessler 1985) is an example of a generalised measurement in the spin factor space. This relatively simple example should show the most important properties of generalised measurements and allows a comparison with the SGSM which is, in the spin factor space, one of the rare examples of a standard measurement. Two points, very important in other circumstances, will not be discussed in this paper. The first one is the wavepacket reduction and the second one is the proper quantum description of the scattering process (e.g., as given in Kraus (1983) ch 5). Instead, we will simply adopt the projection postulate for the standard quantum measurement and, in the scattering process in the MSSM, no description of the target will be made.

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The paper is organised as follows. In next section we give a brief description of the SGSM and some general remarks on predictive and retrospective aspects of measurements. Section 3 contains a simplified description of the MSSM. In § 4 we consider the Mott spin measurement in C^2, pointing out the difference between its predictive and retrospective contents. In § 5 an explicit example of the MSSM in terms of generalised measurements is given. Section 6 concludes the paper.

2. Stern–Gerlach spin measurement

The aim of this section is to give the notation and a simplified description of the SGSM. Furthermore, the SGSM is made use of for a discussion on the retrospective and predictive aspects of measurements.

The appropriate Hilbert space is a composite one $H = H_0 \otimes H_s$; H_0 is the orbital factor space and $H_s = C^2$ is the spin ($s = \frac{1}{2}$) factor space. Circumflexed capital letters will denote operators over a Hilbert space. In particular, projectors are denoted by $\hat{P}$; $\hat{P}^+ = \hat{P}^2 = \hat{P}$, and states by $\hat{W}$; $\hat{W} \geqslant 0, \mathrm{tr}(\hat{W}) = 1$. The set of all states is a convex set having pure states, $\hat{W} = \hat{P}$, as extremal points. In the case of $s = \frac{1}{2}$ the set of states is a ball which is in one-to-one correspondence to the set of polarisation vectors $P = \{P_x, P_y, P_z\}$ where $|P| = (P_x^2 + P_y^2 + P_z^2)^{1/2} \leqslant 1$. States described by $|P| = 1$ are pure states, while $|P| = 0$ describes the unpolarised state $\hat{W} = \frac{1}{2}\hat{I}$. This is the well known 'Poincaré sphere' of polarisations (Bloore 1976).

In a Stern–Gerlach set-up oriented, for example, along the y axis assuming that the initial beam of spin $s = \frac{1}{2}$ particles travel along the z axis, the inspected beam splits in two parts in accordance with its initial polarisation. If an initial state is defined by

$$|f_i\rangle = |f_{0i}\rangle \otimes |s_i\rangle$$

an interaction with a magnetic field gives

$$|f\rangle \sim \langle + | s_i\rangle |f_{0+}\rangle \otimes |+\rangle + \langle - | s_i\rangle |f_{0-}\rangle \otimes |-\rangle.$$

After the localisation, i.e. after the wavepacket reduction, the observed state is

$$\hat{W}_f = |\langle s_i|+\rangle|^2 |f_{0-}\rangle\langle f_{0-}| \otimes |+\rangle\langle +1| + |\langle s_i|-\rangle|^2 |f_{0-}\rangle\langle f_{0-}| \otimes |-\rangle\langle -|. \tag{1}$$

The two orbital parts $|f_{0+}\rangle$ and $|f_{0-}\rangle$ overlap negligibly and the experimental set-up should ascertain their orthogonality, not only that $\langle f_{0+}|f_{0-}\rangle \sim 0$, but also that

$$\int |f_{0+}(r)||f_{0-}(r)| \, \mathrm{d}v \sim 0.$$

The spin states in (1) are also orthogonal and the SGSM may be accepted as a paradigm for a standard quantum measurement in the spin factor space H_s. A description of this process in H_s is given by 'projection postulate' (von Neumann 1955)

$$\hat{W}_f = \sum_k \hat{P}_k \hat{W}_i \hat{P}_k = \sum_k w_k \hat{W}_k \tag{2}$$

where $\hat{W}_i$ is an initial state, $\{\hat{P}_k\}$ are eigenprojectors of the measured observable and $\hat{W}_f$ is the final state. We give the following reasons for accepting the SGSM as a standard measurement:

(i) a value of the measured observable may be assigned to almost every system (in this case by means of different positions), and

(ii) measurements are, in principle, repeatable.

In the set of spin states, SGSM is described by the orthogonal projection of the initial state on the line connecting the eigenprojectors of the measured spin component. Each single system, after the measurement, is either in the state $\hat{P}_{x+}$ or in $\hat{P}_{x-}$ (assuming that the Stern–Gerlach set-up is oriented along the x axis). However, the result of the measurement, performed over an ensemble in an initial state $\hat{W}_i$, is the convex combination

$$\hat{W}_f = \mathrm{tr}(\hat{P}_{x+}\hat{W}_i)\hat{P}_{x+} + \mathrm{tr}(\hat{P}_{x-}\hat{W}_i)\hat{P}_{x-}$$

which coincides with the orthogonal projection of $\hat{W}_i$ onto the line defined by $(\hat{P}_{x+}, \hat{P}_{x-})$. In terms of polarisation vectors, the initial state $P = \{P_x, P_y, P_z\}$ is mapped into the after-measurement state $P' = \{P_x, 0, 0\}$. This is represented in figure 1.

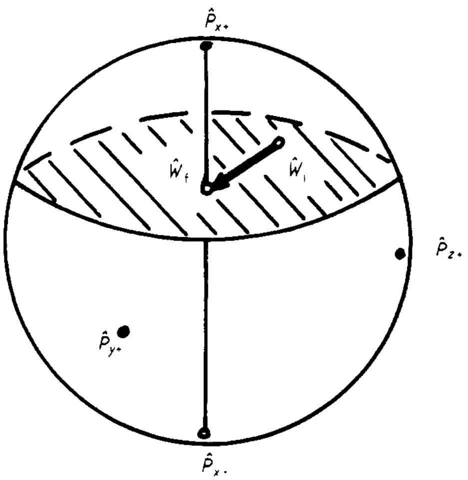

Figure 1. SGSM in set of states for $s = \frac{1}{2}$. States from the shaded area are admissible initial states $\hat{W}_i$ for the final state W_f.

It is easy to infer, from equation (2), that SGSM gives a precise description of the final state, independently of the knowledge on the initial state. This is valid both for an ensemble and for a single system. In the case of retrospection, the set of admissible initial states is the circle orthogonal on $(\hat{P}_{x+}, \hat{P}_{x-})$ at the point $\hat{W}_f$, i.e. any state from that circle will, in the measurement of $\hat{s}_x$, give the same result, W_f. This is valid for an ensemble of systems; for a single system, occurrence of, for example, $\hat{P}_{x+}$ in the measurement means that the initial state of the system can be any state different from $\hat{P}_{x-}$.

In this paper it will be important to differentiate between the predictive and the retrospective aspects of measurements. The predictive aspect concerns one's ability to describe the after-measurement state of the inspected ensemble or system from the results of measurement, and the retrospective one concerns the ability to infer the pre-measurement, initial, state of a system or an ensemble from the results of a measurement. For example, the classical measurement is both retrospective and predictive while the standard quantum measurement can be made predictive (for any complete measurement) both for an ensemble and for a single system. On the other hand, standard quantum measurement is retrospective, but only with respect to the measured observable and it is inadequate for the determination of the pre-measurement state. A careful differentiation between these aspects will be important for a proper understanding of generalised measurements.

3. Mott scattering spin measurement

The SGSM works, for example for uncharged particles, but if one wants to measure, for example the spin of an electron, the only way is to use the Mott scattering spin measurement (MSSM). Here we give a description of the MSSM following Kessler (1985).

In Mott scattering (polarised beam, unpolarised target), due to the spin-orbit coupling the following situation occurs: the initial state in $H_0 \otimes H_s$ is defined by

$$|f_\mathrm{i}\rangle = |f_{0\mathrm{i}}\rangle \otimes |s_\mathrm{i}\rangle.$$

We assume that

$$\langle r, \theta, \phi | f_\mathrm{i}\rangle \sim \exp(\mathrm{i}kz) \qquad |s_\mathrm{i}\rangle \sim P = \{P_x, 0, P_z\} \qquad P_x^2 + P_z^2 = 1 \tag{3}$$

(a complete initial polarisation in the (x, z) plane), where r, θ and ϕ are spherical coordinates: θ is measured from the $+z$ axis; ϕ from the $+x$ axis in the (x, y) plane and the distance from the origin which is placed in the interaction region is measured by r.

As has already been stated, the z axis is given by the initial momentum of particles, while the x axis is defined *a posteriori* so that the initial polarisation vector always lies in the (x, z) plane.

For a single particle, scattered and localised near the point (r_0, θ_0, ϕ_0), the spin state is

$$\hat{W}_{s\mathrm{f}} \sim \hat{S}(\theta, \phi)\, \hat{W}_{s\mathrm{i}} \hat{S}^+(\theta, \phi) \tag{4}$$

where $\hat{S}$ is the scattering matrix (cf Kessler (1985), equations (3.59) and (3.61))

$$\hat{S}(\theta, \phi) = \begin{pmatrix} f(\theta) & -g(\theta)\exp(-\mathrm{i}\phi) \\ g(\theta)\exp(\mathrm{i}\phi) & f(\theta) \end{pmatrix} \tag{5}$$

written in the basis in which $\hat{s}_z$ is diagonal while f and g are the scattering amplitudes. One may introduce four parameters, I, S, T and U,

$$I(\theta) = |f|^2 + |g|^2 \qquad S(\theta) = \mathrm{i}(fg^* - f^* g)/I$$
$$T(\theta) = (|f|^2 - |g|^2)/I \qquad U(\theta) = (fg^* + f^* g)/I \tag{6}$$

and after the scattering, localisation of a particle near the point (r_0, θ_0, ϕ_0) occurs with probability (Kessler (1985), equation (3.70))

$$p(r_0, \theta_0, \phi_0) \sim I(\theta_0)[1 - S(\theta_0)\sin(\phi_0) P_x] = \mathrm{tr}(\hat{S}\hat{W}_{s\mathrm{i}}\hat{S}^+). \tag{7}$$

New polarisation parameters, characterising the final spin state (equation (4)) for a single particle are (Kessler (1985), equation (3.75))

$$P'_x = \frac{P_x + U(\theta_0)P_z\cos(\phi_0) - (1 - T(\theta_0))\cos^2(\phi_0)P_x - S(\theta_0)\sin(\phi_0)}{[1 - S(\theta_0)P_x\sin(\phi_0)]}$$

$$P'_y = \frac{(T(\theta_0)P_x - P_x)\sin(2\phi_0) + 2(S(\theta_0)\cos(\phi_0) + U(\theta_0)P_z\sin(\phi_0))}{2[1 - S(\theta_0)P_x\sin(\phi_0)]} \tag{8}$$

$$P'_z = \frac{T(\theta_0)P_z - U(\theta_0)P_x\cos(\phi_0)}{[1 - S(\theta_0)P_x\sin(\phi_0)]}.$$

S, T, U ($S^2 + T^2 + U^2 = 1$) and I (from (6)) are dependent on the kind of the target, the energy of particles and the observed angle θ.

In order to obtain the MSSM the following preparations are necessary. An unpolarised beam, containing $N(N \gg 1)$, particles is scattered from the target. At the distance r_0 in the ring-shaped region between θ_0 and $\theta_0 + \Delta\theta$ which we denote by $R(r_0, \theta_0)$ N_1 particles are detected. These particles should be evenly distributed with respect to ϕ (because of (7)). The next step is to use a completely polarised beam, for example along the x axis. This time, again due to (7), particles are scattered asymmetrically with respect to ϕ. The observed asymmetry is a maximal one because of a completely polarised initial beam. The 'calibration' can be completed after N_1 particles are detected in the ring $R(r_0, \theta_0)$.

For an incident beam, having an unknown polarisation, scattering of N_1 particles in the ring $R(r_0, \theta_0)$ allows the determination of the spin projection of the initial polarisation on the (x, y) plane. The observed asymmetry in the distribution of N_1 particles detected inside the ring $R(r_0, \theta_0)$ gives data on the orientation of the spin projection in the (x, z) plane (orthogonal to the line connecting the maximum and the minimum of the particle distribution over the ring $R(r_0, \theta_0)$) and on the amount of polarisation in the (x, y) plane (through the comparison with data on a completely polarised and the completely unpolarised initial beam).

Obviously, MSSM essentially differs from the standard quantum measurement giving the spin projection in the chosen plane, which is not an observable. Another point is that a careful distinction must be made between the retrospective part (equation (7)) and the predictive part (equations (4) and (7)) for the MSSM.

4. Description of MSSM in C^2

In this section we give a description of MSSM as it occurs in the spin factor space H_s. In § 4.1 we assume the initial spin state to be known in advance while in § 4.2 we assume that the initial spin state is unknown.

4.1. MSSM *with a known initial state*

An approximate description of the after-measurement state for the collection of particles scattered in the ring $R(r_0, \theta_0)$ is given by

$$\hat{W}_f = \sum_k [1 - SP_x\sin(\phi_k)]\,|f_k\rangle\langle f_k| \otimes \hat{W}_{sf}(\phi_k) \tag{9}$$

where $|f_k\rangle = |f(r_0, \theta_0, \phi_k)\rangle$ are almost orthogonal vectors having wavefunctions which are as near as possible to $\delta(\boldsymbol{r} - (r_0, \theta_0, \phi_k))$ 'describing' the point (r_0, θ_0, ϕ_k). The spin state of a particle detected near the point (r_0, θ_0, ϕ_k) is

$$\hat{W}_{sf}(\phi_k) = \tfrac{1}{2}\begin{pmatrix} 1+P'_z & P'_x+\mathrm{i}P'_y \\ P'_x-\mathrm{i}P'_y & 1-P'_z \end{pmatrix} = \hat{S}\hat{W}_{si}\hat{S}^{+}/\mathrm{tr}(\hat{S}\hat{W}_{si}\hat{S}^{+}) \tag{10}$$

where P'_j, $j = x, y, z$, are given by (8) and $\{\phi_k = 2k\pi/n\}$ gives an appropriate equipartition of the ring $R(r_0, \theta_0)$.

The approximation mentioned above follows from the assumed 'orthogonal' decomposition in the orbital space by means of $\{|f_k\rangle\}$ satisfying $\langle f_k|f_m\rangle \sim \delta_{km}$ and $\Sigma_k |f_k\rangle\langle f_k| \sim |r_0\rangle\langle r_0| \otimes \hat{P}(\theta_0, \theta_0+\Delta\theta) \otimes \hat{I}_\phi$ but it may be accepted without serious difficulties. This almost orthogonal decomposition in H_0 induces a non-orthogonal decomposition in the spin factor space, by means of $\hat{S}$, which is a very important feature of generalised measurements. However, for a proper description one must take the trace of (9) over H_0 in order to obtain the accurate after-measurement spin state.

If the initial spin state is

$$\hat{W}_{si} = \tfrac{1}{2}\begin{pmatrix} 1+P_z & P_x \\ P_x & 1-P_z \end{pmatrix} \qquad P_x^2 + P_z^2 = 1$$

the final spin state for the collection of particles scattered in the ring $R(r_0, \theta_0)$ is

$$\hat{W}_{sf} = \sum_k (1 - SP_x(\phi_k)\,\hat{W}_{sf}(\phi_k)) = \tfrac{1}{2}\begin{pmatrix} 1+P_zT & \tfrac{1}{2}P_x(1+T) \\ \tfrac{1}{2}P_x(1+T) & 1-P_zT \end{pmatrix}. \tag{11}$$

One should notice that the set of pure states

$$P_x^2 + P_y^2 + P_z^2 = 1$$

is, by (11), mapped onto the ellipsoid

$$[2P'_x/(1+T)]^2 + [2P'_y/(1+T)]^2 + (P'_z/T)^2 = 1.$$

Equations (10) and (11) give a dynamical map in the set of states for the spin $s = \frac{1}{2}$ case, performed by the MSSM. How this looks in the set of spin states is shown in figure 2.

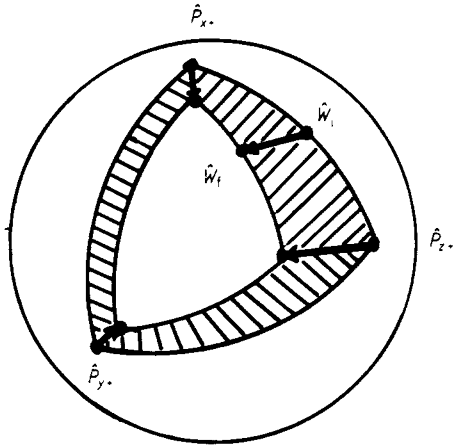

Figure 2. Predictive MSSM in set of states. Ensemble change of state for $T = \frac{1}{2}$. The set of pure states is mapped on the ellipsoid.

4.2. MSSM *with unknown initial state*

As already stated, the retrospective result of MSSM is given by (7); it determines the projection of the initial spin state on the (x, y) plane. Chosing the x axis *a posteriori*, the retrospection of the initial spin state is

$$\hat{W}_s^{\mathrm{r}} = \tfrac{1}{2}\begin{pmatrix} 1 & P_x \\ P_x & 1 \end{pmatrix} \tag{12}$$

where P_x was obtained from (7). The set of admissible pre-measurement states is the segment orthogonal to the (x, y) plane at the point $\hat{W}_s^{\mathrm{r}}$. The advantage of the MSSM for retrospection compared to the SGSM is obvious.

On the other hand, concerning the prediction, the after-measurement state for a MSSM, for which (12) is the retrospected state, can be any state given by (11) assuming that P_z is a parameter satisfying $0 \leq P_z \leq (1 - P_x^2)^{1/2}$. Therefore the MSSM gives an accurate prediction only if the initial state is known; otherwise the prediction is incomplete. What has been said is valid for ensembles; for a single system MSSM gives neither the prediction nor retrospection. Again this can be represented in the set of spin states (figure 3).

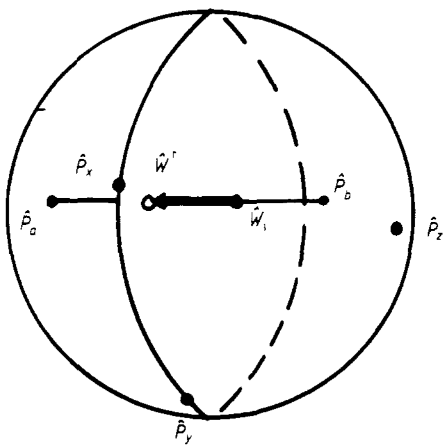

Figure 3. Retrospective MSSM in set of states. The set of admissible initial states for $\hat{W}^{\mathrm{r}}$ is represented by the segment $\{\hat{P}_a, \hat{P}_b\}$. $\hat{W}^{\mathrm{r}} = \frac{1}{2}(\hat{P}_a + \hat{P}_b)$.

It is an interesting fact that data about the initial polarisation, which are obtained from the MSSM, are equivalent to the data obtained from two SGSM performed along the x and y axes. As a consequence, the MSSM is nearer to a state determination procedure (Ivanovic 1981) than SGSM is.

5. Generalised measurement formulation

In this section an explicit formulation of the MSSM as a predictive generalised measurement in spin space is given.

Generalised quantum measurement is usually represented by

$$\hat{W}_{\mathrm{f}} = \sum_k \hat{B}_k^{+} W_{\mathrm{i}} \hat{B}_k \tag{13}$$

where $\Sigma_k \hat{B}_k\hat{B}_k^+ = \hat{I}$ is the corresponding decomposition of the identity. Equation (13) simply reduces to (2) in the case when $\{\hat{B}_k\}$ is a set of mutually orthogonal projectors, i.e. an orthogonal decomposition of the identity. For a given decomposition $\{\hat{A}_k\}$, $\hat{A}_k \geqslant 0$, $\Sigma_k \hat{A}_k = \hat{I}$ factorisations $\hat{A}_k = \hat{B}_k\hat{B}_k^+$ are non-unique; also different decompositions may give the same change of a state.

Another kind of identity decomposition is the continuous one:

$$\hat{I} = \int \hat{A}(t)\,\mathrm{d}t.$$

Perhaps the best known example is that of a coherent state decomposition. The aim of this section is to offer an explicit realisation of a continuous decomposition of the identity which corresponds to the MSSM.

The most natural decomposition is already given through the set of scattering matrices $\hat{S}(\theta, \phi)$ (equation (5)). In terms of S, T, U and I (equation (6)) and up to a normalisation constant

$$\hat{S}(\theta, \phi) \sim \begin{pmatrix} (U-\mathrm{i}S)/(1-T) & \exp(-\mathrm{i}\phi) \\ -\exp(-\mathrm{i}\phi) & (U-\mathrm{i}S)/(1-T) \end{pmatrix}$$

while

$$\hat{S}\hat{S}^+ = I\begin{pmatrix} 1 & -\mathrm{i}S\exp(-\mathrm{i}\phi) \\ \mathrm{i}S\exp(\mathrm{i}\phi) & 1 \end{pmatrix}$$

written in the basis in which $\hat{s}_z$ is diagonal.

Chosing $\theta = \theta_0$, with proper normalisation the desired decomposition in the ring $R(r_0, \theta_0)$ is given by

$$\hat{I}_{H_s} = \int_U^{2\pi} \hat{S}\hat{S}^+(\mathrm{d}\phi/2\pi I). \tag{14}$$

Hence the single-system spin state scattered near (r_0, θ_0, ϕ_0) is given by $\hat{W}_\mathrm{f} = \hat{S}\hat{W}_\mathrm{i}\hat{S}^+$ while the ensemble state for the particles scattered inside $R(r_0, \theta_0)$ is

$$\hat{W}_\mathrm{f} = \int_0^{2\pi} \hat{S}\hat{W}_\mathrm{i}\hat{S}^+(\mathrm{d}\phi/2\pi I). \tag{15}$$

In the complete decomposition of the identity, i.e. for all values of θ, the decomposition given by (14) should be repeated for every value of θ, with different $\hat{S}$ and an appropriate normalisation. One should notice that this is valid both for pure and mixed initial spin states.

The ensemble change of state (15) can be obtained in a number of ways and we give one more example:

$$\hat{B}_1 = \begin{pmatrix} [1-2(a^2+c^2)]^{1/2} & 0 \\ 0 & [1-2(b^2+c^2)]^{1/2} \end{pmatrix}$$

$$\hat{B}_2 = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \qquad \hat{B}_3 = \begin{pmatrix} a & -c \\ -c & b \end{pmatrix}$$

where a, b and c are real parameters satisfying

$$a, b, c > 0 \qquad ab > c \qquad (a^2+c^2) < \tfrac{1}{2} \qquad (b^2+c^2) < \tfrac{1}{2}.$$

Nevertheless, the continuous decomposition (14) allows the most satisfactory interpretation for the description of single systems.

It is possible to identify (15) as a non-selective measurement assuming that the measurement apparatus is composed of the target and of a photoplate placed inside the ring $R(r_0, \theta_0)$, due to the fact that only after the localisation of a particle does one know the exact scattering matrix $\hat{S}(\theta_0, \phi)$ which has been activated. Also, non-occurrence of a localisation near (r_0, θ_0, ϕ) does not mean that $\hat{W}_f - \hat{S}\hat{W}_i\hat{S}^+$ has occurred, as will be the case for the standard measurement.

6. Conclusions

We conclude this paper with several remarks. The first one is that all calculations are given for the rest frames of the scattered particles. To obtain an ensemble for which (11) is valid, one must perform a number of preparatory position measurements in the ring $R(r_0, \theta_0)$. Collimation of all particles along a single direction without altering its spin will give the desired ensemble.

Compared to the SGSM the MSSM is better for retrospective measurements while the opposite is true for predictive measurements; it is useful to repeat the main features of the MSSM:

(i) no value of an observable (in the spin space) can be assigned to a single system as a measurement result if the initial state is unknown;

(ii) there exists a minimal number of particles, for each set-up which must be scattered in the desired region in order to obtain an accurate result, and

(iii) there is no repeatability.

These properties should be valid for the majority of generalised measurements.

Finally, a better understanding of generalised measurements which are a class of dynamical maps may improve our insight into a proper quantum description of the measurement process.

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